Fault Detection and Isolation of Linear Roesser Systems with Stochastic Communications

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***Abstract*— This paper addresses the problem of the fault detection and isolation of the two-dimensional (2D) linear Roesser systems with stochastic communication. Stochastic data transmission from the plant to the observer through the network is considered to reduce the required bandwidth of the communication network significantly. In this regard, the fault detection and isolation problem, while being robust with respect to the disturbances, is modeled as two H_∞ and a H_- optimization problems. The overall design approach of the observer with stochastic data transmission is proposed as a linear optimization problem with linear matrix inequality (LMI) constraints to get the best robust performance in fault detection and isolation while maintaining the stability of the observer. Finally, the effectiveness of the developed robust fault detection and isolation observer with stochastic reduced output data transmission is shown through some simulations.**

***Keywords*: Fault Diagnosis, Roesser Systems, Stochastic Communications, Two-Dimensional Systems.**

I. Introduction

INDUSTRIAL processes, including thousands of sensors, actuators, and control loops, are continuously monitored. Suppose an abnormality occurs in any of the system components, affecting the system's normal function. In that case, this problem will be informed to the operator by an audio or visual alarm. All system components are sensitive to faults that may disrupt the normal function of the process or even cause damage and dangerous situations. The increasing complexity of industrial processes, besides increasing demand for safety and better quality and efficiency, makes us diagnose the faults of the systems more accurately and efficiently. There are other undesired inputs like noise and different kinds of disturbances in the systems, and their effects can be mistaken as the effects of the faults.

The monitoring system should be able to distinguish between the faults and other unknown inputs like noise and disturbances. To this end, many fault detection and isolation approaches have been presented. Some of these methods do not rely on the model of the plant [1]. These methods use the system's inputs and measured outputs for the fault diagnosis [2]. The other classes of fault diagnosis methods are based on available models of the plant and do fault detection, isolation, or estimation depending on their approach [3].

In modern industrial control systems, communication networks are used for the data and control commands to the plant and information of the plant to the control room due to many control and monitoring loops. So that these networks have become an inseparable part of today’s control and monitoring systems. Scheduling, delays, and data packet losses are some of the challenges for network control and monitoring systems. [4] proposes a fault detection approach based on the minimum variance benchmark for linear networked systems with time-varying delays and missing data packets. A fault-tolerant method for compensating the actuator fault in the distributed network control systems has been presented in [5], while the time delays, quantization errors, and data packet losses are considered. The problem of network fault tolerant control of the nonlinear systems with unknown time-varying sensor faults has been addressed in [6].

Reducing the data transmission rate through the network is one of the other important issues in networked control and monitoring systems. In this regard, event-triggered data transmission is one of the most popular approaches. In the event-triggered framework, a new data packet is sent through the network only if a particular condition is satisfied, which reduces the required network bandwidth [7].

Data transmission in a stochastic manner or based on protocols like Round-Robin is the other approach that has gotten attention in research papers on networked control and monitoring systems [8-10]. All of these approaches are based on the principle of reducing the data transmissions as much as possible to reduce the implementation costs.

Two-dimensional (2D) systems are another class of dynamical systems with two independent variables despite the one-dimensional systems that only have time as their independent variable and ordinary differential equations that can describe their model. The most famous classes of 2D systems are Roesser and Fornasini-Marchesini models (The Fornasini-Marchesini model is divided into the first and second types) [11]. They have various applications in different areas. Image processing is one of their application fields in that the horizontal and vertical indices of the image are presented by the two independent variables of the 2D systems [12]. Iterative and learning control systems are another field of application for 2D systems. One independent variable represents time, and the other represents the repetition index in the control system [13]. Modeling the partial differential equation (PDE) systems is another application of the 2D systems in that one of the independent variables represents time, and the other represents the spatial index in the control system [14]. Regarding the mentioned applications of 2D systems in the Iterative and learning control and PDE systems, there are many papers on the control [15-17] and monitoring [14, 18-20] of the 2D systems.

The communication networks have gotten attention on the control and observer design of the 2D systems for a reason similar to the conventional ODE systems. A robust iterative controller for batch processes with time delays and data packet losses has been presented in [21]. The controller design problem using an observer for the Roesser systems with packet dropouts has been investigated in [22]. In [23], has generalized the sliding mode control of the Roesser system in the event-triggered framework. The problem of the networked simultaneous fault detection and isolation and robust control of the Roesser systems with different kinds of disturbances has been addressed in [24]. For the first time, the event-triggered control of the Fornasini-Marchesini systems has been investigated in [25]. Event-triggered control of the Markov jump 2D systems has been proposed in [26]. In the event-triggered approach, some conditions must be monitored for the decision about sending or not sending the data packets all the time. The data transmission rates reduce depending on how strict these conditions are. As the data transmission rates reduce, less network bandwidth is required. The main problem with the event-triggered approach is always checking some specific conditions to decide on sending or not sending the data packets. Furthermore, the exact required network bandwidth cannot be predicted. Another approach for reducing the required bandwidth of the communication network is to send just a portion of the data packets at every sampling time without checking any conditions. This data transmission can obey a specific protocol like Round-Robin or can be in a stochastic manner. There are some results on this kind of communication for the 2D systems in [27-31]. A sliding mode controller for the Roesser systems has been presented in [27]. In [28], a robust output controller for the Fornasini-Marchesini systems with stochastic communications has been proposed. The filtering problem of the Fornasini-Marchesini systems has been addressed in [30, 31] with communication-based on the Round-Robin protocol.

In the mentioned papers on control and observer design with stochastic and Round-Robin data transmissions, Fornasini-Marchesini's second model systems are mainly investigated. No paper exists on the Roesser systems' fault diagnosis (fault detection and isolation) with stochastic data transmission. Therefore, this paper aims to design a robust observer for the linear Roesser systems with a stochastic data transmission mechanism in the presence of disturbances to detect and isolate faults in the system. The system output is transmitted to the observer through the network, and only one of the system outputs will be sent randomly at each sampling time. A dedicated residual for each fault is considered, and the bank of residuals will be designed to have the most sensitivity to their related faults and be robust concerning the disturbances and other faults as much as possible. The observer design method for each of the two fault sensitivity and robustness with respect to the disturbances are presented separately. Then, a unified design method based on a linear constrained optimization problem is proposed to achieve the two goals simultaneously.

The rest of the paper is organized as follows. The model of the system, observer structure, and some mathematical preliminaries will be presented in section II. Section III presents the stochastic data transmission mechanism, some theorems about the observer design for robustness with respect to the disturbances, and fault detection and isolation. After that, a unified design approach is proposed through a constrained linear optimization problem. Section IV provides some simulations to show the effectiveness of the proposed fault detection and isolation scheme. Finally, the conclusion is presented in section V.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider the following linear Roesser system:

$$\begin{bmatrix} x_h(i+1,j) \\ x_v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_h(i,j) \\ x_v(i,j) \end{bmatrix} + \begin{bmatrix} B_{f1} \\ B_{f2} \end{bmatrix} f(i,j) + \begin{bmatrix} B_{d1} \\ B_{d2} \end{bmatrix} d(i,j)$$

$$y(i,j) = \begin{bmatrix} y_1(i,j) \\ y_2(i,j) \\ \vdots \\ y_p(i,j) \end{bmatrix} = [C_1 \quad C_2]x(i,j) + D_f f(i,j) + D_d d(i,j)$$

where $x_h(i,j) \in \mathbb{R}^{n_h}$, $x_v(i,j) \in \mathbb{R}^{n_v}$, $y(i,j) = [y_1(i,j) \quad \dots \quad y_p(i,j)]^T \in \mathbb{R}^p$, $d(i,j) \in \mathbb{R}^{n_d}$, and $f(i,j) \in \mathbb{R}^{n_f}$ are the state variables (including the horizontal state variable vector $x_h(i,j)$ and vertical state variable vector $x_v(i,j)$), the measured outputs, the disturbances, and the fault inputs, respectively. The matrices of the above Roesser system have the appropriate dimensions regarding the mentioned signals. The model of the mentioned Roesser system is considered as below, hereafter:

$$\mathfrak{S}x(i,j) = Ax(i,j) + B_f f(i,j) + B_d d(i,j) \qquad (1)$$
$$y(i,j) = Cx(i,j) + D_f f(i,j) + D_d d(i,j)$$

where $\mathfrak{S}$ is the shift operator and defined as $\mathfrak{S}x(i,j) = [x_h^T(i+1,j) \quad x_v(i,j+1)]^T$, and $x(i,j) = [x_h^T(i,j) \quad x_v^T(i,j)]^T \in \mathbb{R}^n$. $x_h(0,j)$ and $x_v(i,0)$ are the boundary conditions of system (1). Furthermore, the faults and disturbances are energy bounded.

The main aim of this paper is to design a robust observer and a bank of residuals (a dedicated residual for each fault) to detect and isolate the occurred faults in the system (1) while being robust with respect to the disturbances. To this end, consider the following observer:

$$\mathfrak{S}\hat{x}(i,j) = A_f\hat{x}(i,j) + L\bar{y}(i,j)$$
$$r(i,j) = R\big(\bar{y}(i,j) - C\hat{x}(i,j)\big) \qquad (2)$$

where $r(i,j) \in \mathbb{R}^{n_f}$, $\hat{x}(i,j) = [\hat{x}_h^T(i,j) \quad \hat{x}_v^T(i,j)]^T \in \mathbb{R}^n$ $(n = n_h + n_v)$ are the residuals and the state variables of the observer, respectively. Since $\hat{x}(i,j) \in \mathbb{R}^n$, then $A_f \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times p}$, and $R \in \mathbb{R}^{n_f\times p}$. Furthermore, $\bar{y}(i,j)$ is the latest measured output that is available. The goal is to detect the occurrence of the faults and isolate them between a group of possible faults. Therefore, a dedicated residual is assigned for each fault.

Assumption 1: The boundary conditions of the system (1) satisfy the following inequality:

$$\lim_{N\to\infty} E\left\{\sum_{k=0}^{N}(\|x_v(k,0)\|^2 + \|x_h(0,k)\|^2)\right\} < \infty$$

where $\|x_v(k,0)\|^2 = x_v^T(k,0)x_v(k,0)$ and $\|x_h(0,k)\|^2 = x_h^T(0,k)x_h(0,k)$.

Lemma 1: Given $V(i,j) = x^T(i,j)Px(i,j)$ as the Lyapunov function for the 2D system (1) with $P = \begin{bmatrix} P_h & 0 \\ 0 & P_v \end{bmatrix}$, the following equation holds:

$$\sum_{i=0}^{k_h}\sum_{j=0}^{k_v} \Delta V(i,j) = \sum_{i=1}^{k_h}[V_v(i,k_v+1) - V_v(i,0)] + \sum_{j=1}^{k_v}[V_h(k_h+1,j) - V_h(0,j)] \qquad (3)$$

Where:

$$V_h(i,j) = x_h^T(i,j)P_h x_h(i,j)$$
$$V_v(i,j) = x_v^T(i,j)P_v x_v(i,j) \qquad (4)$$

Proof: Regarding $V(i,j) = x^T(i,j)Px(i,j)$ and $P = \begin{bmatrix} P_h & 0 \\ 0 & P_v \end{bmatrix}$, the Lyapunov function $V(i,j)$ can be rewritten as:

$$V(i,j) = x_h^T(i,j)P_h x_h(i,j) + x_v^T(i,j)P_v x_v(i,j) = V_h(i,j) + V_v(i,j)$$

Therefore:

$$\Delta V(i,j) = \Delta V_h(i,j) + \Delta V_v(i,j) \qquad (5)$$

where:

$$\Delta V_h(i,j) = V_h(i+1,j) - V_h(i,j) = x_h^T(i+1,j)P_h x_h(i+1,j) - x_h^T(i,j)P_h x_h(i,j) \qquad (6)$$

$$\Delta V_v(i,j) = V_v(i,j+1) - V_v(i,j) = x_v^T(i,j+1)P_v x_v(i,j+1) - x_v^T(i,j)P_v x_v(i,j) \qquad (7)$$

Using (6), (8) can be obtained. In a similar way (9) is derived using (7). Then, equation (3) can easily be concluded by (5), (8), and (9).

III. MAIN RESULTS

The structure of the robust fault isolation observer is shown in (2), where $\bar{y}(i,j)$ is used instead of $y(i,j)$ in it. Actually, only one of p measured outputs is sent through the network due to bandwidth limitations of the communication network. A zero-order hold is used for other outputs to keep their previously available values. Therefore, $\frac{p-1}{p}$ percent will be saved in the bandwidth of

$$\begin{aligned}\sum_{i=0}^{k_h}\sum_{j=0}^{k_v}\Delta V_h(i,j) &= \sum_{i=0}^{k_h}\sum_{j=0}^{k_v}[V_h(i+1,j)-V_h(i,j)]\\ &= [V_h(1,0)-V_h(0,0)]+[V_h(1,1)-V_h(0,1)]+[V_h(1,2)-V_h(0,2)]+\cdots\\ &+[V_h(1,k_v)-V_h(0,k_v)]+[V_h(2,0)-V_h(2,0)]+[V_h(2,1)-V_h(1,1)]+\cdots\\ &+[V_h(2,k_v)-V_h(1,k_v)]+\cdots+[V_h(k_h+1,0)-V_h(k_h,0)]+\cdots\\ &+[V_h(k_h+1,k_v)-V_h(k_h,k_v)] = \sum_{j=0}^{k_v}[V_h(k_h+1,j)-V_h(0,j)] \end{aligned} \tag{8}$$

$$\begin{aligned}\sum_{i=0}^{k_h}\sum_{j=0}^{k_v}\Delta V_v(i,j) &= \sum_{i=0}^{k_h}\sum_{j=0}^{k_v}[V_v(i,j+1)-V_v(i,j)]\\ &= [V_v(0,1)-V_v(0,0)]+[V_v(0,2)-V_v(0,1)]+\cdots+[V_v(0,k_v+1)-V_v(0,k_v)]\\ &+[V_v(1,1)-V_v(1,0)]+[V_v(1,2)-V_v(1,1)]+\cdots+[V_v(1,k_v+1)-V_v(1,k_v)]+\cdots\\ &+[V_v(k_h,1)-V_v(k_h,1)]+\cdots+[V_v(k_h,k_v+1)-V_v(k_h,k_v)]\\ &= \sum_{i=0}^{k_h}[V_v(i,k_v+1)-V_v(i,0)] \end{aligned} \tag{9}$$

the network. For example, in a system with two outputs ($p=2$), 50 percent for the bandwidth of the network will be saved. This amount goes higher as the number of outputs increases. Considering "i" as the spatial index and "j" as the temporal index, the output $\bar{y}(i,j)$ is modelled by:

$$\bar{y}(i,j) = \beta_{h(j)}y(i,j) + \left(I-\beta_{h(j)}\right)\bar{y}(i,j-1) \tag{10}$$

where:

$$\beta_{h(j)} = \begin{bmatrix}\delta(h(j)-1) & 0 & 0 & 0\\ 0 & \delta(h(j)-2) & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \delta(h(j)-p)\end{bmatrix}$$

$h(j)$ indicates which of the p outputs are sent to the observer through the network ($1\le h(j)\le p$). Therefore, $\beta_{h(j)}$ is a diagonal matrix with only one non-zero element related to the selected output through p outputs for sending to the observer. This output is selected randomly in each sampling time, and its updated data will be sent to the observer while the other outputs preserve their previous values using a zero-order hold. The probability of choosing each output is θ_k for $1\le k\le p$ and considered to be known. Therefore:

$$P\{h(j)=k\} = P\left\{\beta_{h(j)} = \begin{bmatrix}0&0&0&0&0\\0&\ddots&0&0&0\\0&0&1&0&0\\0&0&0&\ddots&0\\0&0&0&0&0\end{bmatrix}\right\} = \theta_k \tag{11}$$

In each sampling period, one of the outputs is surely sent to the observer through the network ($\sum_{k=1}^{q}\theta_k = 1$). Fig. 1 shows a schematic overview of the plant alongside observer and stochastic data transmission.

A new augmented system with the new state variable $X(i,j) = [x_h(i,j)\quad \hat{x}_h(i,j)\quad x_v(i,j)\quad \hat{x}_v(i,j)\quad \bar{y}(i,j-1)]^T$ can be derived as:

$$\begin{aligned}\mathfrak{S}X(i,j) &= \tilde{A}(j)X(i,j) + \tilde{B}_f(j)f(i,j) + \tilde{B}_d(j)d(i,j)\\ r(i,j) &= \tilde{C}(j)X(i,j) + \bar{D}_f(j)f(i,j) + \bar{D}_d(j)d(i,j)\end{aligned} \tag{12}$$

where:

$$\tilde{A}(j) = \Pi\begin{bmatrix}A & 0 & 0\\ L\beta_{h(j)}C & A_f & L\left(I-\beta_{h(j)}\right)\\ \beta_{h(j)}C & 0 & \left(I-\beta_{h(j)}\right)\end{bmatrix}\Pi^T$$

$$\tilde{B}_f(j) = \Pi\begin{bmatrix}B_f\\ L\beta_{h(j)}D_f\\ \beta_{h(j)}D_f\end{bmatrix}, \tilde{B}_d(j) = \Pi\begin{bmatrix}B_d\\ L\beta_{h(j)}D_d\\ \beta_{h(j)}D_d\end{bmatrix}$$

$$\tilde{C}(j) = \left[R\beta_{h(j)}C \quad -RC \quad R\left(I-\beta_{h(j)}\right)\right]\Pi^T, \bar{D}_f(j) = R\beta_{h(j)}D_f, \bar{D}_d(j) = R\beta_{h(j)}D_d$$

$$\Pi = \begin{bmatrix}I_{n_h} & 0 & 0 & 0 & 0\\ 0 & 0 & I_{n_h} & 0 & 0\\ 0 & I_{n_v} & 0 & 0 & 0\\ 0 & 0 & 0 & I_{n_v} & 0\\ 0 & 0 & 0 & 0 & I_p\end{bmatrix}$$

The aim is for the designed observer (2) to be stable and for the residual $r(i,j)$ is sensitive to the faults while being robust to disturbances. The augmented system (12) contains the stochastic matrix $\beta_{h(j)}$. Therefore, an exclusive stability and a robustness concept will be defined.

Definition 1: The 2D system (12) with zero input is mean-square stable if, for any boundary conditions satisfying Assumption 1, the following condition holds:

$$\lim_{i+j\to\infty} E\{\|x(i,j)\|^2\} = 0$$

Definition 2: The 2D system (12) is robustly stable with respect to the energy-bounded input $d(i,j)$ in the H_∞ sense with attenuation level γ if the following inequality holds while having zero boundary conditions:

$$E\{\|r\|_2^2\} \le \gamma^2\|d\|_2^2$$

where:

$$E\{\|r\|_2^2\} = E\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} r^T(i,j)r(i,j)\right\}$$

$$\|d\|_2^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d^T(i,j)d(i,j)$$

As mentioned earlier, we want to design a robustly stable observer to successfully detect and isolate the faults of the 2D system (1). To this end, the matrix coefficients "L"and "R" should be determined such that:

1) The state vector $X(i,j)$ is stable in the min-square sense.
2) The following performance indices are satisfied for the robustness of the residual $r(i,j)$ with respect to the disturbance $d(i,j)$ while being sensitive to the fault $f(i,j)$:

$$\text{(I)}E\{\|r\|_2^2\} \leq \gamma_d^2\|d\|_2^2,\ \text{(II)}\inf\left(\frac{\|Jf\|_2}{\|f\|_2}\right) > 1,$$
$$\text{(III)}E\{\|r-Jf\|_2^2\} \leq \gamma_f^2\|f\|_2^2$$

The performance index (I) is related to the robustness of the residual $r(i,j)$ with respect to the disturbance $d(i,j)$ to reduce its effect by the attenuation level γ_d. H_- performance index for fault detection and isolation consists of the performance indices (II) and (III) to ensure the fault sensitivity of the residuals and achieve fault isolation simultaneously. "J" is a filter that can be dynamic or static. The simplest choice to achieve fault isolation is a diagonal matrix. Furthermore, the performance index (II) is related to the minimum sensitivity of the residuals with respect to the faults. On the other hand, the performance index (III) forces the residuals to track the filtered fault signal "$Jf(i,j)$". The diagonal structure of the matrix "J" causes the first residual is dedicated to the first fault, the second residual is dedicated to the second fault, and so on.

The matrices A_f and L have an impact on the stability of the augmented system (12), and the robustness properties of the observer (2) regarding the disturbance and the faults for satisfying the performance indices (I) and (III). On the other hand, the matrix R can only manipulate the robustness properties and help to satisfy the performance indices (I) and (III). The matrices A_f and L have no specific structure, but the matrix R should be full column rank to reveal the effects of the fault inputs in the residuals completely.

The observer design process for achieving stability while satisfying the performance indices (I), (II), and (III) are developed separately and then merged to obtain a unified approach.

A. Analysis and design regarding the disturbance input

In this section, the analysis and design process of the observer (2) for robustness with respect to the disturbance input is presented while maintaining stability. The sufficient condition for the stability and robustness analysis of the augmented system with respect to the disturbance $d(i,j)$ is stated in the following theorem.

Theorem 1: Consider the 2D system (1) and the observer (2) with the stochastic output data transmission (10). Given the matrix coefficients L, A_f, R and the attenuation level γ_d, the augmented system (12) is min-square stable and the H_∞ performance index (I) is satisfied under zero boundary conditions if there exist positive definite matrix $P \in \mathbb{R}^{(2n+p)\times(2n+p)}$ such that the following matrix inequality holds:

$$\begin{bmatrix} \tilde{A}^T P\tilde{A} + \tilde{C}^T\tilde{C} - P & \tilde{A}^T P\tilde{B}_d + \tilde{C}^T\overline{D}_d \\ * & \tilde{B}_d^T P\tilde{B}_d + \overline{D}_d^T\overline{D}_d - \gamma_d^2 I \end{bmatrix} < 0 \quad (13)$$

where:

$$\tilde{A} = \Pi\begin{bmatrix} A & 0 & 0 \\ L\Theta C & A_f & L(I-\Theta) \\ \Theta C & 0 & (I-\Theta)\end{bmatrix}\Pi^T,\ \tilde{B}_d = \Pi\begin{bmatrix} B_d \\ L\Theta D_d \\ \Theta D_d\end{bmatrix},$$

$$\tilde{C} = [R\Theta C \quad -RC \quad R(I-\Theta)]\Pi^T, \overline{D}_d = R\Theta D_d,$$

$$P = \begin{bmatrix} P_{h1} & 0 & 0 & 0 & 0 \\ 0 & P_{h2} & 0 & 0 & 0 \\ 0 & 0 & P_{v1} & 0 & 0 \\ 0 & 0 & 0 & P_{v2} & 0 \\ 0 & 0 & 0 & 0 & P_{v3}\end{bmatrix},$$

$$\Theta = \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \theta_q\end{bmatrix} = E\{\beta_{h(j)}\},$$

and $P_{h1} \in \mathbb{R}^{n_h\times n_h}$, $P_{h2} \in \mathbb{R}^{n_h\times n_h}$, $P_{v1} \in \mathbb{R}^{n_v\times n_v}$, $P_{v2} \in \mathbb{R}^{n_v\times n_v}$, and $P_{v3} \in \mathbb{R}^{p\times p}$.

Proof: first, the stability of the augmented system (12) will be shown based on Definition 1. To this end, Z_1, and Z_2 are introduced as:

$$Z_1 = E\{[\mathfrak{S}X(i,j)]^T P(\mathfrak{S}X(i,j))|X(i,j)\}$$
$$Z_2 = X^T(i,j)PX(i,j) \quad (14)$$

Consider the index $J = Z_1 - Z_2$. Substitution of the augmented system (12) in J leads to:

$$\begin{aligned} J &= E\{[\mathfrak{S}X(i,j)]^T P[\mathfrak{S}X(i,j)]|X(i,j)\} \\ &- X^T(i,j)PX(i,j) \\ &= E\left\{\left[\tilde{A}(j)X(i,j)\right]^T P\left[\tilde{A}(j)X(i,j)\right]\middle|X(i,j)\right\} \\ &- X^T(i,j)PX(i,j) \\ &= X^T(i,j)\left[\tilde{A}^T P\tilde{A} - P\right]X(i,j) \\ &= X^T(i,j)\Lambda X(i,j) \end{aligned} \quad (15)$$

Regarding (13), it is obvious that $\Omega < 0$. Therefore, for any $X(i,j)$ it can be concluded:

$$\begin{aligned} \frac{Z_1 - Z_2}{Z_2} &= -\frac{X^T(i,j)(-\Omega)X(i,j)}{X^T(i,j)PX(i,j)} \\ &\leq -\frac{\lambda_{\min}(-\Omega)}{\lambda_{\max}(P)} = \sigma - 1 \end{aligned} \quad (16)$$

where $\sigma = 1 - \left[\frac{\lambda_{\min}(-\Omega)}{\lambda_{\max}(P)}\right]$. Inequality

(16) leads to:

$$\sigma \geq \frac{Z_1}{Z_2} > 0 \tag{17}$$

It is obvious that $\frac{\lambda_{\min}(-\Omega)}{\lambda_{\max}(P)} > 0$ regarding $P > 0$ and $\Omega < 0$. As a result $0 < \sigma < 1$ and independent of $X(i,j)$. Furthermore, $Z_1 \leq \sigma Z_2$ can be concluded by (17). Taking expectation of both sides result in the following:

$$E\{[\mathfrak{S}X(i,j)]^T P[\mathfrak{S}X(i,j)]|X(i,j)\} \leq \sigma X^T(i,j)PX(i,j) \tag{18}$$

By substitution of i and j for 0 through $k+1$ we have:

$$E\{X_v^T(k+1,0)P_vX_v(k+1,0)\} = E\{X_v^T(k+1,0)P_vX_v(k+1,0)\}$$

$$E\left\{[X_h^T(k+1,0) \quad X_v^T(k,1)]P\begin{bmatrix}X_h(k+1,0)\\X_v(k,1)\end{bmatrix}\right\} \leq \sigma E\left\{[X_h^T(k,0) \quad X_v^T(k,0)]P\begin{bmatrix}X_h(k,0)\\X_v(k,0)\end{bmatrix}\right\}$$

$$E\left\{[X_h^T(k,1) \quad X_v^T(k-1,2)]P\begin{bmatrix}X_h(k,1)\\X_v(k-1,2)\end{bmatrix}\right\} \leq \sigma E\left\{[X_h^T(k-1,1) \quad X_v^T(k-1,1)]P\begin{bmatrix}X_h(k-1,1)\\X_v(k-1,1)\end{bmatrix}\right\}$$

$$E\left\{[X_h^T(k-1,2) \quad X_v^T(k-2,3)]P\begin{bmatrix}X_h(k-1,2)\\X_v(k-2,3)\end{bmatrix}\right\} \leq \sigma E\left\{[X_h^T(k-2,2) \quad X_v^T(k-2,2)]P\begin{bmatrix}X_h(k-2,2)\\X_v(k-2,2)\end{bmatrix}\right\}$$

$$\vdots$$

$$E\left\{[X_h^T(1,k) \quad X_v^T(0,k+1)]P\begin{bmatrix}X_h(1,k)\\X_v(0,k+1)\end{bmatrix}\right\} \leq \sigma E\left\{[X_h^T(0,k) \quad X_v^T(0,k)]P\begin{bmatrix}X_h(0,k)\\X_v(0,k)\end{bmatrix}\right\}$$

$$E\{X_h^T(0,k+1)P_hX_h(0,k+1)\} = E\{X_h^T(0,k+1)P_hX_h(0,k+1)\}$$

Summing both sides of these inequalities and two equations will lead to (19). Then, by subsequent usage of inequality (19), (20) will be obtained. Norm properties result in the inequality (21), where $\phi = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$. Let us introduce $\mathcal{X}_k = \sum_{j=0}^{k}\|X(k-j,j)\|^2$. Then, by taking into account (21), we have:

$$E\{\mathcal{X}_0\} \leq \phi E\{\|X_v(0,0)\|^2 + \|X_h(0,0)\|^2\}$$

$$E\{\mathcal{X}_1\} \leq \phi[\sigma E\{\|X_v(0,0)\|^2 + \|X_h(0,0)\|^2\} + E\{\|X_v(1,0)\|^2 + \|X_h(0,1)\|^2\}]$$

$$E\{\mathcal{X}_2\} \leq \phi[\sigma^2 E\{\|X_v(0,0)\|^2 + \|X_h(0,0)\|^2\} + \sigma E\{\|X_v(1,0)\|^2 + \|X_h(0,1)\|^2\} + E\{\|X_v(2,0)\|^2 + \|X_h(0,2)\|^2\}]$$

$$\vdots$$

$$E\{\mathcal{X}_N\} \leq \phi[\sigma^N E\{\|X_v(0,0)\|^2 + \|X_h(0,0)\|^2\} + \sigma^{N-1}E\{\|X_v(1,0)\|^2 + \|X_h(0,1)\|^2\} + \cdots + E\{\|X_v(N,0)\|^2 + \|X_h(0,N)\|^2\}]$$

$$\begin{aligned}
&E\left\{\sum_{j=0}^{k+1}[X_h^T(k+1-j,j) \quad X_v^T(k+1-j,j)]P\begin{bmatrix}X_h(k+1-j,j)\\X_v(k+1-j,j)\end{bmatrix}\right\}\\
&\quad \leq \sigma E\left\{\sum_{j=0}^{k}X^T(k-j,j)PX(k-j,k)\right\} + E\{X_h^T(0,k+1)P_hX_h(0,k+1)\}\\
&\quad + E\{X_v^T(k+1,0)P_vX_v(k+1,0)\}
\end{aligned} \tag{19}$$

$$\begin{aligned}
&E\left\{\sum_{j=0}^{k+1}[X_h^T(k+1-j,j) \quad X_v^T(k+1-j,j)]P\begin{bmatrix}X_h(k+1-j,j)\\X_v(k+1-j,j)\end{bmatrix}\right\}\\
&\quad \leq \sigma^{k+1}E\{X_h^T(0,0)P_hX_h(0,0)\} + E\{X_v^T(0,0)P_vX_v(0,0)\}\\
&\quad + E\left\{\sum_{j=0}^{k}\sigma^j[X_h^T(0,k+1-j) \quad X_v^T(k+1-j,0)]P\begin{bmatrix}X_h(0,k+1-j)\\X_v(k+1-j,0)\end{bmatrix}\right\}\\
&\quad + E\{X_h^T(0,k+1)P_hX_h(0,k+1)\} + E\{X_v^T(k+1,0)P_vX_v(k+1,0)\}\\
&\quad = E\left\{\sum_{j=0}^{k+1}\sigma^j[X_h^T(0,k+1-j) \quad X_v^T(k+1-j,0)]P\begin{bmatrix}X_h(0,k+1-j)\\X_v(k+1-j,0)\end{bmatrix}\right\}
\end{aligned} \tag{20}$$

$$E\left\{\sum_{j=0}^{k+1}\|X(k+1-j,j)\|^2\right\} \leq \phi\sum_{j=0}^{k+1}\sigma^j\{\|X_v(k+1-j,0)\|^2 + \|X_h(0,k+1-j)\|^2\} \tag{21}$$

$$\begin{aligned}
&\sum_{k=0}^{N}E\{\mathcal{X}_k\} \leq \phi(1+\sigma+\cdots+\sigma^N)E\{\|X_v(0,0)\|^2 + \|X_h(0,0)\|^2\}\\
&\quad + \phi(1+\sigma+\cdots+\sigma^{N-1})E\{\|X_v(1,0)\|^2 + \|X_h(0,1)\|^2\} + \cdots\\
&\quad + \phi E\{\|X_v(N,0)\|^2 + \|X_h(0,N)\|^2\}\\
&\quad \leq \phi(1+\sigma+\cdots+\sigma^N)E\{\|X_v(0,0)\|^2 + \|X_h(0,0)\|^2\}\\
&\quad + \phi(1+\sigma+\cdots+\sigma^N)E\{\|X_v(1,0)\|^2 + \|X_h(0,1)\|^2\} + \cdots\\
&\quad + \phi(1+\sigma+\cdots+\sigma^N)E\{\|X_v(N,0)\|^2 + \|X_h(0,N)\|^2\}\\
&\quad = \phi\left(\frac{1-\sigma^N}{1-\sigma}\right)E\left\{\sum_{k=0}^{N}(\|X_v(k,0)\|^2 + \|X_h(0,k)\|^2)\right\}
\end{aligned} \tag{22}$$

The summation of both sides results in (22).

The right-hand side of (22) consists of $\phi\left(\frac{1-\sigma^N}{1-\sigma}\right)$ and $E\{\sum_{k=0}^{N}(\|X_v(k,0)\|^2+\|X_h(0,k)\|^2)\}$. The first term ($\phi\left(\frac{1-\sigma^N}{1-\sigma}\right)$) is obviously bounded. The second term is the same as in Assumption 1, which is taken as a bounded value. This means that $\sum_{k=0}^{N}E\{\mathcal{X}_k\}$ is bounded. If $N\to\infty$, $\sum_{k=0}^{N}E\{\mathcal{X}_k\}$ still remains bounded (because the right-hand side remains bounded). $E\{\mathcal{X}_k\}$ for $k=0$ to $k\to\infty$ is a series of non-negative terms with $\lim_{N\to\infty}\sum_{k=0}^{N}E\{\mathcal{X}_k\}<\infty$. The boundedness of this summation requires that $\lim_{N\to\infty}E\{\mathcal{X}_N\}=0$. Therefore, $\lim_{i+j\to\infty}E\{\|X(i,j)\|^2\}=0$ and the augmented system (12) is min-square stable regarding Definition 1.

We multiply the inequality (13) by $[X^T(i,j)\quad d^T(i,j)]$ and its transpose from left and right, respectively, to show the robustness of the augmented system (12) with respect to the disturbance $d(i,j)$. Considering Θ as the expected value of $\beta_{h(j)}$, we have:

$$E\left\{[X^T(i,j)\quad d^T(i,j)]\Lambda\begin{bmatrix}X(i,j)\\ d(i,j)\end{bmatrix}\right\}<0$$

where:

$$\Lambda=\begin{bmatrix}\Lambda_{11} & \tilde{A}^T(j)P\tilde{B}_d(j)+\tilde{C}^T(j)\bar{D}_d(j)\\ * & \bar{B}_d^T(j)P\bar{B}_d(j)+\bar{D}_d^T(j)\bar{D}_d(j)-\gamma_d^2I\end{bmatrix}$$

$$\Lambda_{11}=\tilde{A}^T(j)P\tilde{A}(j)+\tilde{C}^T(j)\tilde{C}(j)-P$$

Which can be rewritten as:

$$\mathrm{E}\{\Delta V(i,j)+r^T(i,j)r(i,j)-\gamma_d^2d^T(i,j)d(i,j)\}<0 \qquad (23)$$

where:

$$\begin{aligned}\Delta V(i,j)&=\Delta V_h(i,j)+\Delta V_v(i,j)\\&=[V_h(i+1,j)-V_h(i,j)]\\&\quad+[V_v(i,j+1)-V_v(i,j)]\end{aligned}$$

Furthermore:

$$V_h(i,j)=X_h^T(i,j)\begin{bmatrix}P_{h1} & 0\\ 0 & P_{h2}\end{bmatrix}X_h(i,j)$$

$$V_v(i,j)=X_v^T(i,j)\begin{bmatrix}P_{v1} & 0 & 0\\ 0 & P_{v2} & 0\\ 0 & 0 & P_{v3}\end{bmatrix}X_v(i,j)$$

Double summation of both sides of
(23) for $i\ge 0$ and $j\ge 0$ results in (expected value is a linear operator):

$$\mathrm{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}[\Delta V(i,j)+r^T(i,j)r(i,j)-\gamma_d^2d^T(i,j)d(i,j)]\right\}<0 \qquad (24)$$

Under zero boundary conditions and taking into account Lemma 1, it can be concluded that $\mathrm{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta V(i,j)\right\}\ge 0$. As a result:

$$\mathrm{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}r^T(i,j)r(i,j)-\gamma_d^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}d^T(i,j)d(i,j)\right\}<0$$

Then:

$$E\{\|r(i,j)\|_2^2\}<\gamma_d^2\|d(i,j)\|_2^2$$

■

The stability and robustness analysis of the augmented system (12) was investigated in Theorem 1. The following theorem is about sufficient LMI conditions for the design of the robust observer (2) with stochastic output transmission.

Theorem 2: Consider the 2D system (1) and the observer (2) with the stochastic output data transmission (10). Given the attenuation level γ_d, the augmented system (12) is min-square stable and the H_∞ performance index (I) is satisfied under zero boundary conditions if there exist matrices $S\in\mathbb{R}^{n\times p}$, $M\in\mathbb{R}^{n\times n}$ and the positive definite matrix $Q\in\mathbb{R}^{(2n+p)\times(2n+p)}$ such that the following LMI holds:

$$\begin{bmatrix}-Q & 0 & \bar{A}^TQ & \bar{C}^T\\ * & -\gamma_d^2I & \bar{B}_d^TQ & \bar{D}_d^T\\ * & * & -Q & 0\\ * & * & * & -I\end{bmatrix}<0 \qquad (25)$$

where:

$$Q=\Pi^{\mathrm{T}}P\Pi=\begin{bmatrix}P_{h1} & 0 & 0 & 0 & 0\\ 0 & P_{v1} & 0 & 0 & 0\\ 0 & 0 & P_{h2} & 0 & 0\\ 0 & 0 & 0 & P_{v2} & 0\\ 0 & 0 & 0 & 0 & P_{v3}\end{bmatrix}$$

$$=\begin{bmatrix}P_1 & 0 & 0\\ 0 & P_2 & 0\\ 0 & 0 & P_3\end{bmatrix}$$

$$\bar{A}^TQ=\begin{bmatrix}A^TP_1 & C^T\Theta S^T & C^T\Theta P_3\\ 0 & M^T & 0\\ 0 & (I-\Theta)S^T & (I-\Theta)P_3\end{bmatrix},$$

$$\bar{B}_d^TQ=[B_d^TP_1\quad D_d^T\Theta S^T\quad 0]$$

$$\bar{C}=[R\Theta C\quad -RC\quad R(I-\Theta)]$$

The matrix $\bar{D}_d$ is the same as Theorem 1. Furthermore, the matrix coefficients L and A_f can be obtained by $A_f=P_2^{-1}M$, and $L=P_2^{-1}S$ (R is a part of $\bar{C}$ and is obtained by solving (25)).

Proof: Applying Schur lemma on (13) leads to:

$$\begin{bmatrix}-P & 0 & \tilde{A}^T & \tilde{C}^T\\ * & -\gamma_d^2I & \tilde{B}_d^T & \bar{D}_d^T\\ * & * & -P^{-1} & 0\\ * & * & * & -I\end{bmatrix}<0 \qquad (26)$$

After multiplying (26) by $diag(I,I,P,I)$ from the right and left sides, we have:

$$\begin{bmatrix}-P & 0 & \tilde{A}^TP & \tilde{C}^T\\ * & -\gamma_d^2I & \tilde{B}_d^TP & \bar{D}_d^T\\ * & * & -P & 0\\ * & * & * & -I\end{bmatrix}<0 \qquad (27)$$

Then the LMI (25) will be obtained by multiplying the inequality (27) by $diag(\Pi^{\mathrm{T}}, I, \Pi^{\mathrm{T}}, I)$ and $diag(\Pi, I, \Pi, I)$ from the left and right sides, respectively. This shows the sufficiency of LMI (25) for the matrix inequality (13). Furthermore, the sufficiency of (13) for the stability and robustness of the augmented system (12) with respect to the disturbance $d(i,j)$ is shown in Theorem 1. Therefore, the sufficiency of the LMI (25) for the stability and robustness of the augmented system (12) is proved.

■

B. Analysis and design for fault detection and isolation

This section presents the analysis and design process of the observer (2) for fault detection and is presented while maintaining stability. In the following theorem, the sufficient conditions for the stability analysis of the augmented system (12) and the simultaneous satisfaction of H_- performance index (II) for fault detection, and H_∞ performance index (III) for the fault isolation are presented.

Theorem 3: Consider the 2D system (1) and the observer (2) with the stochastic output data transmission (10). Given the matrix coefficients $J > I$, L, A_f, R and the attenuation level γ_f, the augmented system (12) is min-square stable and the H_∞ performance index (II) and H_- performance index (II) are satisfied under zero boundary conditions if there exist positive definite matrix $P \in \mathbb{R}^{(2n+p)\times(2n+p)}$ such that the following matrix inequality holds:

$$\begin{bmatrix} \tilde{A}^T P \tilde{A} + \tilde{C}^T \tilde{C} - P & \tilde{A}^T P \tilde{B}_f + \tilde{C}^T \overline{D}_f \\ * & \tilde{B}_f^T P \tilde{B}_f + \overline{D}_f^T \overline{D}_f - \gamma_f^2 I \end{bmatrix} < 0 \qquad (28)$$

where:

$$\tilde{B}_d = \Pi \begin{bmatrix} B_f \\ L\Theta D_f \\ \Theta D_f \end{bmatrix}, \overline{D}_f = R\Theta D_f - J$$

The matrices $\tilde{A}$, $\tilde{C}$, and P are defined similarly to Theorem 1.

Proof: It is obvious that $J > I$ satisfies the performance index (II). The proof of satisfaction of performance index (III) by inequality (28) is similar to Theorem 1, which is omitted for the sake of brevity.

■

The following theorem presents sufficient conditions for the observer and residual design for fault detection and isolation.

Theorem 4: Consider the 2D system (1) and the observer (2) with the stochastic output data transmission (10). Given the attenuation level γ_f, the augmented system (12) is min-square stable and the H_- performance index (II) and the H_∞ performance index (III) are satisfied under zero boundary conditions if there exist matrices $S \in \mathbb{R}^{n\times p}$, $M \in \mathbb{R}^{n\times n}$ and the positive definite matrix $Q \in \mathbb{R}^{(2n+p)\times(2n+p)}$ such that the following LMIs hold:

$$J - I > 0 \qquad (29)$$

$$\begin{bmatrix} -Q & 0 & \bar{A}^T Q & \bar{C}^T \\ * & -\gamma_f^2 I & \bar{B}_f^T Q & \overline{D}_f^T \\ * & * & -Q & 0 \\ * & * & * & -I \end{bmatrix} < 0 \qquad (30)$$

where:

$$\bar{B}_f^T Q = \begin{bmatrix} B_f^T P_1 & D_f^T \Theta S^T & 0 \end{bmatrix}$$

The other matrices are defined similarly to Theorem 1. Furthermore, the matrix coefficients L and A_f can be obtained by $A_f = P_2^{-1} M$, and $L = P_2^{-1} S$ (R is a part of $\bar{C}$ and is obtained by solving (25)).

Proof: About the satisfaction of the H_- performance index (II) is discussed in Theorem 3. The performance index (III) is similar to the H_∞ performance index (I). Therefore, the sufficiency of the LMI (30) for the matrix inequality (28) can be shown similarly. As a result, the mean-square stability of the augmented system (12) and the ability of the observer (2) for fault detection and isolation can be proved, which is omitted for the sake of brevity.

■

In Theorem 4 and Theorem 2, the observer design with stochastic communications (to decrease the required bandwidth for the communication network) was investigated for achieving fault detection isolation and robustness with respect to the disturbances while being augmented stable, separately. Both theorems should be considered simultaneously to achieve satisfactory results. In the performance index (I), the effects of the disturbance on the residual decreases as the attenuation level γ_d is reduced. Similarly, the fault isolation is done better if the performance index (III) is satisfied with smaller values of γ_f, and the interferences of the faults to their unrelated residuals are reduced (for example, the first residual gets the biggest impact from the first fault and so much smaller effects for the other faults). Therefore, it is desired to reduce the attenuation levels γ_d and γ_f as much as possible. This reduction in the attenuation levels is possible to a certain level and depends on the system matrices. Therefore, the following optimization problem is used to minimize both of the γ_d and γ_f simultaneously:

$$\min_{L, A_f, R} \alpha_1 \Gamma_d + \alpha_2 \Gamma_f \\ s.t. (25), (29), and\ (30)\ hold \qquad (31)$$

where α_1 and α_2 are the weighting coefficients. Furthermore, $\Gamma_f = \gamma_f^2$ and $\Gamma_d = \gamma_d^2$ to make the optimization problem (31) linear. As the proportion $\frac{\alpha_2}{\alpha_1}$ increases, the importance of the robustness with respect to the disturbance $d(i,j)$ is more intensified and vice versa (As the proportion $\frac{\alpha_1}{\alpha_2}$ increases, the importance of fault isolation is intensified).

IV. Simulations

In this section, the performance of the proposed robust fault detection and isolation filter will be evaluated through some simulations.

Example 1: Consider the following partial differential equation describing a heat process:

$$\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - 4T(x,t) + u(x,t) \quad (32)$$

The derivative terms can be approximated by:

$$\frac{\partial T(x,t)}{\partial x} \approx \frac{T(i,j) - T(i-1,j)}{\Delta x}$$
$$\frac{\partial T(x,t)}{\partial t} \approx \frac{T(i,j+1) - T(i,j)}{\Delta t}$$

Regarding the mentioned approximations, the Roesser model of the system (32) with zero input will be:

$$\mathfrak{S}X(i,j) = \begin{bmatrix} 0 & 1 \\ \frac{\Delta t}{\Delta x} & 1 - \frac{\Delta t}{\Delta x} - 4\Delta t \end{bmatrix} X(i,j)$$

where $X(i,j) = \begin{bmatrix} T(i-1,j) \\ T(i,j) \end{bmatrix}$. Assuming $dt = 0.1$, $dx = 0.4$:

$$A = \begin{bmatrix} 0 & 1 \\ 0.25 & 0.35 \end{bmatrix}$$

The other matrices of the system are considered as:

$$B_d = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, B_f = \begin{bmatrix} 0.5 & 0.03 \\ 0.6 & 0.1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0.2 & 1 \end{bmatrix},$$
$$D_d = \begin{bmatrix} 0.09 \\ 0.2 \end{bmatrix}, D_f = \begin{bmatrix} 1 & 0.1 \\ 0.2 & 1 \end{bmatrix}$$

The disturbance input is considered as $d(i,j) = \frac{j}{30}\sin\left(\frac{2\pi i}{20}\right)$ and the fault inputs are taken as:

$$f(i,j) = \begin{cases} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & (2 < i < 10) \&\& (3 < j < 10) \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & (15 \le i \le 25) \&\& (j \ge 15) \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & otherwise \end{cases}$$

The weighting coefficients $\alpha_1 = \alpha_2 = 1$ are chosen. The probability of sending each of the two outputs of the plant through the network are considered as $\theta_1 = \theta_2 = 0.5$. The robust observer (2) is designed using the constrained optimization problem (31). The results of the fault detection and isolation of the observer is depicted in Fig. 2. Furthermore, the token passing between the outputs of the plant for the data transmission through the network is presented in Fig. 3. It is shown in Fig. 2 that both of the faults are detected and isolated in their related fault, and the other fault effect is weakened. Furthermore, the effect of the disturbance input is weakened successfully. The plant has two outputs. Therefore, the access token to the network should toggle between two values. In Fig. 3, the blue (yellow) squares are related to the update points of the first (second) output in different spatial and temporal indices. The probability of passing tokens to each of the outputs is equal to 0.5, verifiable by the almost equal numbers of the yellow and blue squares in Fig. 3. The robust performance bounds of the observer (2) for this example are reported in Table 1.

Example 2: Consider the following partial differential equation describing a gas absorption or water stream heating process:

$$\frac{\partial^2 S(x,t)}{\partial x \partial t} = a_1 \frac{\partial S(x,t)}{\partial t} + a_2 \frac{\partial S(x,t)}{\partial x} + a_3 S(x,t) + bu(x,t) \quad (33)$$

The model of the system

(33) using $W(x,t) = \frac{\partial S(x,t)}{\partial t} - a_2 S(x,t)$ can be rewritten as:

$$\begin{bmatrix} \frac{\partial W(x,t)}{\partial x} \\ \frac{\partial S(x,t)}{\partial t} \end{bmatrix} = \begin{bmatrix} a_1 & a_1 a_2 + a_3 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} W(x,t) \\ S(x,t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u(x,t)$$

TABLE 1
Robust performance bounds for example 1

Transition Probability	γ_d	γ_f
$\theta_1 = \theta_2 = 0.5$	0.521	0.754

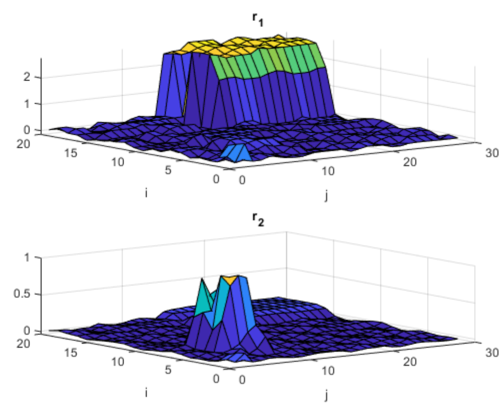

Fig. 2. The residuals for the fault detection and isolation

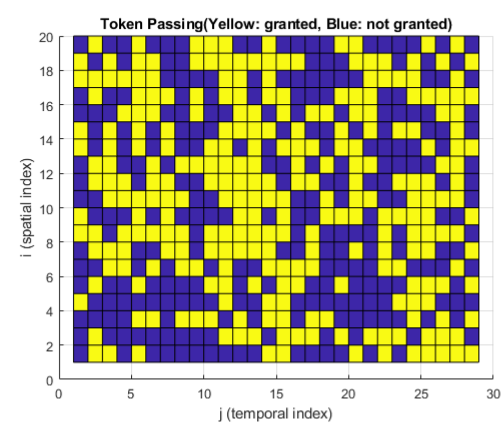

Fig. 3. Token passing between the two outputs of the plant

Similar to example 1, its Roesser model without input is obtained as:

$$\mathfrak{S}X(i,j) = \begin{bmatrix} 1 + a_1\Delta x & (a_1 a_2 + a_3)\Delta x \\ \Delta t & 1 + a_2\Delta t \end{bmatrix} X(i,j)$$

Assuming $dt = 0.1$, $dx = 0.4$, $a_1 = -0.5$, $a_2 = a_3 = -1$:

$$A = \begin{bmatrix} 0.8 & -0.2 \\ 0.1 & 0.9 \end{bmatrix}$$

The other matrices of the system are considered as:

$$B_d = \begin{bmatrix} 0.5 & 0.2 \\ 0 & -0.05 \end{bmatrix}, B_f = \begin{bmatrix} 0.05 & 0.7 \\ 0.8 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.2 & 0.3 \\ -0.8 & 0.4 \end{bmatrix},$$

$$D_d = \begin{bmatrix} 0.01 & 0 \\ 0.05 & -0.03 \end{bmatrix}, D_f = \begin{bmatrix} 1 & 0.03 \\ 0.08 & 1 \end{bmatrix}$$

The disturbance input is considered as $d(i,j) = \frac{j}{30}\sin\left(\frac{2\pi i}{20}\right)$ and the fault inputs are taken as:

$$f(i,j) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & (2 < i < 10)\&\&(3 < j < 10) \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & (15 \le i \le 25)\&\&(j \ge 15) \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & otherwise \end{cases}$$

The weighting coefficients $\alpha_1 = \alpha_2 = 1$ are chosen. The probability of sending each of the two outputs of the plant through the network are considered as $\theta_1 = 0.3$ and $\theta_2 = 0.7$. The robust observer (2) is designed similarly to example 1. The results of the fault detection and isolation of the observer are depicted in Fig. 4. Furthermore, the token passing between the outputs of the plant for the data transmission through the network is presented in Fig. 5. Similar to example 1, both of the faults are detected and isolated in their related fault, and the other fault effect is weakened. Furthermore, the effect of the disturbance input is weakened successfully. The plant has two outputs, and the probability of passing the token to each output equals 0.3 and 0.7, respectively. This effect is verifiable in Fig. 5. The robust performance bounds of the observer (2) for this example are reported in Table 2.

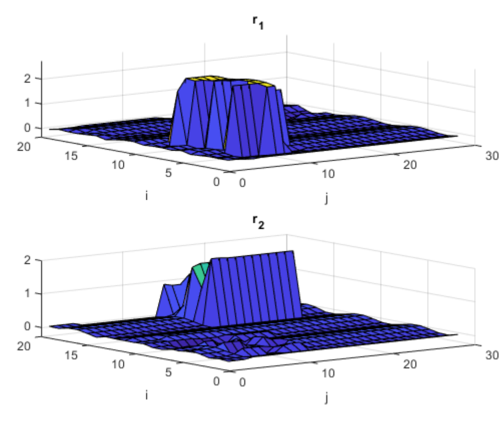

Fig. 4. The residuals for the fault detection and isolation

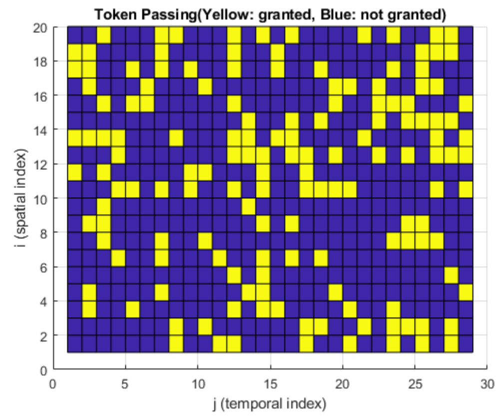

Fig. 5. Token passing between the two outputs of the plant

TABLE 2
Robust performance bounds for example 2

Transition Probability	γ_d	γ_f
$\theta_1 = 0.3, \theta_2 = 0.7$	0.935	0.627

V. Conclusion

This paper proposes a robust observer for the fault detection and isolation of the linear Roesser with stochastic output transmission through the communication network. At each sampling time, only one of the plant's outputs is selected randomly to be sent through the network and updated on the observer side, while the other outputs keep their previous values. This leads to the reduction of the required data packet transmissions through the network. The problem of fault detection and isolation is modeled as a H_-/H_∞ optimization problem. Furthermore, the robustness of the bank of the residuals with respect to the disturbances is modeled as a H_∞ optimization problem. The sufficient conditions for satisfying the mentioned optimization problems are separately stated as LMIs. An observer design approach for simultaneously fulfilling all objectives (observer stability, fault detection and isolation, and robustness concerning the disturbances) is presented as a constrained linear optimization problem. Finally, the performance of the proposed observer is evaluated through some simulations. The developed robust fault detection and isolation observer is uses a linear Roesser model without delay, and the communication links are assumed to be perfect. The incorporation of delay terms and data packet losses are considered for future studies.

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